2016/17 MATH2230B/C Complex Variables with Applications
Problems in HW 3
Due Date on 9 Mar 2017

All the problems are from the textbook, Complex Variables and Application (9*th* edition).

1 P.133

For the functions f and contours C in Exercise 8, use parametric representations for C or legs of C to evaluate

$$\int_C f(z)\,\mathrm{d}z.$$

8. $f(z)$ is the principal branch

$$z^{a-1} = \exp\big[(a-1)\mathrm{Log}z\big] \quad (|z| > 0, -\pi < \mathrm{Arg}z < \pi)$$

of the power function z^{a-1}, where a is a nonzero real number and C is the positively oriented circle of radius R about the origin.

11. Let C denote the semicircular path

$$C = \{z : |z| = 2, \mathrm{Re}(z) \geq 0\}.$$

Evaluate the integral of the function $f(z) = \overline{z}$ along C using the parametric representation

(a) $z = 2e^{i\theta} \quad \Big(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\Big)$;

(b) $z = \sqrt{4-y^2} + iy \quad (-2 \leq y \leq 2)$.

2 P.139

5. Let C_R be the circle $|z| = R$ $(R > 1)$, described in the counterclockwise direction. Show that

$$\left|\int_{C_R} \frac{\mathrm{Log}z}{z^2}\,\mathrm{d}z\right| < 2\pi\left(\frac{\pi + \ln R}{R}\right),$$

and then use l'Hospital's rule to show that the value of this integral tends to zero as R tends to infinity.

6. Let C_ρ denote a circle $|z| = \rho$ $(0 < \rho < 1)$ oriented in the counterclockwise direction and suppose that $f(z)$ is analytic in the disk $|z| \leq 1$. Show that if $z^{-1/2}$ represents any particular branch of that power of z, then there is a nonnegative constant M independent of ρ such that

$$\left|\int_{C_\rho} z^{-1/2} f(z)\,\mathrm{d}z\right| \leq 2\pi M\sqrt{\rho}.$$

Thus show that the value of the integral here approaches 0 as ρ tends to 0.

3 P.147

5. Show that
$$\int_{-1}^{1} z^i \, \mathrm{d}z = \frac{1+e^{-\pi}}{2}(1-i).$$
where the integrand denotes the principal branch
$$z^i = \exp(i\mathrm{Log}z) \quad (|z| > 0. -\pi < \mathrm{Arg}z < \pi)$$
of z^i and where path of integration is any contour from $z = -1$ to $z = 1$ that except for its end points lies above the real axis.

4 P.159

2. Let C_1 denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 1, y = \pm 1$ and let C_2 be the positively oriented circle $|z| = 4$. With the aid of the corollary in Sec. 53, point out why
$$\int_{C_1} f(z) \, \mathrm{d}z = \int_{C_2} f(z) \, \mathrm{d}z$$
when
   (a) $f(z) = \frac{1}{3z^2+1}$;
   (b) $f(z) = \frac{z+2}{\sin(z/2)}$;
   (a) $f(z) = \frac{z}{1-e^z}$.

4. Use the following method to derive the integration formula
$$\int_0^\infty e^{-x^2} \cos 2bx \, \mathrm{d}x = \frac{\sqrt{pi}}{2} e^{-b^2} \quad (b > 0).$$
   (a) Show that the sum of the integrals of e^{-z^2} along the lower and upper horizontal legs of the rectangular path $\{x = \pm a, y = 0 \mathrm{or} y = b\}$ can be written
$$2\int_0^a e^{-x^2} \, \mathrm{d}x - 2e^{b^2} \int_0^a e^{-x^2} \cos 2bx \, \mathrm{d}x$$
   and that the sum of the integrals along the vertical legs on the right and left can be written
$$ie^{-a^2} \int_0^b e^{y^2 - i2ay} \, \mathrm{d}x - ie^{a^2} \int_0^b e^{y^2 - i2ay} \, \mathrm{d}y.$$
   Thus with the aid of the Cauchy-Coursat theorem, show that
$$\int_0^a e^{-x^2} \cos 2bx \, \mathrm{d}x = e^{-b^2} \int_0^a e^{-x^2} \, \mathrm{d}x + e^{-(a^2+b^2)} \int_0^b e^{y^2} \sin 2ay \, \mathrm{d}y.$$

(b) By accepting the fact that

$$\int_0^\infty e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}$$

and observing that

$$\left|\int_0^b e^{y^2}\sin 2ay\,\mathrm{d}y\right| \le \int_0^b e^{y^2}\,\mathrm{d}y$$

obtain the desired integration formula by letting a tend to infinity in the equation at the end of part (a).

5 P.170

2. Find the value of the integral of $g(z)$ around the circle $|z-i|=2$ in the positive sense when

   (a) $g(z) = \frac{1}{z^2+4}$;

   (b) $g(z) = \frac{1}{(z^2+4)^2}$.

3. Let C be the circle $|z|=3$ described in the positive sense. Show that if

$$g(z) = \int_C \frac{2s^2-s-2}{s-z}\,\mathrm{d}s \quad (|z|\neq 3)$$

   then $g(2) = 8\pi i$. What is the value of $g(z)$ when $|z|>3$?

4. Let C be any simple closed contour described in the positive sense in the z plane and write

$$g(z) = \int_C \frac{s^3+2s}{(s-z)^3}\,\mathrm{d}s.$$

   Show that $g(z) = 6\pi i z$ when z is inside C and that $g(z)=0$ when z is outside.